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AXISYMMETRIC PLASTIC FLOW OF AN IDEALLY CONNECTED MEDIUM WITH FRICTION*

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A new approach is proposed to solving the boundary value problems of a flow corresponding to the sides of piecewise-smooth conditions of plasticity, based on introducing the function of characteristic directions which satisfies a quasilinear second-order hyperbolic equation. Systems of equations are studied describing the plastic flow of an ideally rigid-plastic medium obeying the generalized Coulomb-Mohr condition. It is established that the systems of equations are hyperbolic, and relations on the characteristics are obtained. The possible discontinuities in the stresses and displacement velocities along certain curves in the meridional plane are studied. Analgoues of the variational principles are obtained for the rigid-plastic medium in question.

1. Yield surface.

We adopt

$$\alpha(\sigma_i + \sigma_j) + \beta\sigma_k + |\sigma_i - \sigma_j| = 2k \qquad (1.1)$$
$$i \neq j \neq k;\ i, j, k = 1, 2, \varphi$$

as the generalization of the known piecewise smooth yield surfaces in the space of principal stresses /1-7/.

It can be shown that satisfying the inequalities $\alpha - 1 \leqslant \beta \leqslant \alpha + 1$ represents the sufficient condition for (1.1) to correspond to a convex, six-sided pyramid with apex at the point $\sigma_1 = \sigma_2 = \sigma_\varphi = 2k/(2\alpha + \beta)$ and the axis equally inclined to the axes of the coordinates σ_i $(i = 1, 2, \varphi)$ in the space of principal stresses. Fig.1 shows the intersection of the pyramid surface by the plane $\sigma_1+\sigma_2+\sigma_\varphi=0$. The coefficient α should be associated with $\sin\varphi^\circ$ (φ° is the angle of internal friction); β is the coefficient of lateral pressure (tension) /8/ taking into account the effect of the mean principal stress on the strength of the medium (it depends on the porosity and compaction), and k is the coefficient of adhesion of the medium.

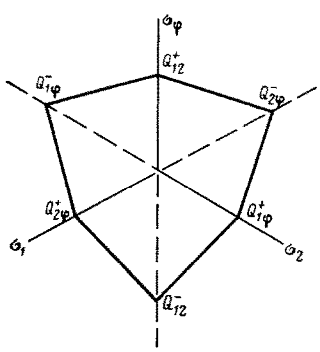

Fig.1

In (1.1) let us change to the stress tensor components $\sigma_r, \sigma_z, \tau_{rz}, \sigma_\varphi$ in r, φ, z axes

$$A(\sigma_r + \sigma_z) + B\sigma_\varphi + C\sqrt{\Sigma} = 2k \qquad (1.2)$$
$$\Sigma = (\sigma_r - \sigma_z)^2 + 4\tau_{rz}^2$$

where A, B, C are determined depending on the type of sides. Following the notation of Fig.1, we have

$$A = \alpha,\quad B = \beta,\ C = 1 \qquad (1.3)$$

for the sides $Q_{2\varphi}^+Q_{1\varphi}^-$ $(\sigma_1 > \sigma_\varphi > \sigma_2)$, $Q_{2\varphi}^-Q_{1\varphi}^+$ $(\sigma_2 > \sigma_\varphi > \sigma_1)$, and

$$A = (\alpha + \beta + \varkappa)/2,\ B = \alpha - \varkappa,\ C = [\varkappa(\alpha - \beta) + 1]/2 \qquad (1.4)$$
$$\varkappa = 1,\ \sigma_2 > \sigma_\varphi;\ \varkappa = -1,\ \sigma_2 < \sigma_\varphi$$

for the sides $Q_{1\varphi}^-Q_{12}^+$ $(\sigma_\varphi > \sigma_1 > \sigma_2)$, $Q_{1\varphi}^+Q_{12}^-$ $(\sigma_2 > \sigma_1 > \sigma_\varphi)$. For the sides $Q_{12}^+Q_{2\varphi}^-$ $(\sigma_\varphi > \sigma_2 > \sigma_1)$, $Q_{12}^-Q_{2\varphi}^+$ $(\sigma_1 > \sigma_2 > \sigma_\varphi)$ we have the relations (1.4) where in the expression for $\varkappa$, we must replace σ_2 by σ_1.

Any of the edges $Q_{12}^+, Q_{2\varphi}^-, \ldots, Q_{1\varphi}^-$ (Fig.1) can be described as an intersection of the adjacent sides

$$A(\sigma_r + \sigma_z) + B\sigma_\varphi + C\sqrt{\Sigma} = 2k \qquad (1.5)$$
$$A_1(\sigma_r + \sigma_z) + B_1\sigma_\varphi + C_1\sqrt{\Sigma} = 2k$$

Here A, A_1, B, B_1, C, C_1 are found from (1.3), (1.4) in accordance with the condition of contiguity.

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We note that condition (1.1) generalizes the piecewise-smooth yield surfaces in the space of principal stresses, known in the theory of ideal plasticity. Thus, when $\alpha = \beta = 0$, we have the Tresca-Saint Venant plasticity prism /1, 5-7/; $\alpha = \sin \varphi^\circ$, $\beta = 0$ is the Coulomb-Mohr condition /2/; $\alpha = \beta$ is the proper Drucker pyramid /3/ and $\beta = \alpha \pm 1$ are three-sided Haythornthwaite pyramids /4/.

2. The defining equations of flow for a side. Let us introduce the Levi variables $\sigma = (\sigma_1 + \sigma_2)/2, \tau = |\sigma_1 - \sigma_2|/2, \Psi$

$$\sigma_r = \sigma + \tau \cos 2\Psi, \quad \sigma_z = \sigma - \tau \cos 2\Psi, \quad \tau_{rz} = \tau \sin 2\Psi \tag{2.1}$$

(Ψ is the angle between the first principal direction in the r, z plane and the r axis, σ_1, σ_2 are the principal stress components in the meridional plane). Using further (1.2) as the plastic potential, we obtain the following expressions for the deformation velocities $\varepsilon_r, \varepsilon_z, \varepsilon_\varphi, \gamma_{rz}$:

$$\begin{aligned} &\varepsilon_r = \partial u/\partial r = \lambda (A + C \cos 2\Psi), \quad \varepsilon_z = \partial v/\partial z = \lambda (A - C \cos 2\Psi) \\ &\varepsilon_\varphi = u/r = \lambda B, \quad \gamma_{rz} = 2\lambda C \sin 2\Psi \end{aligned} \tag{2.2}$$

Here u, v are the velocities of displacement along the r, z axes respectively, and λ is a non-negative multiplier.

Further, if we eliminate λ from (2.2) and σ_φ from the equations of equilibrium using (1.2) in the latter process, and use the variables (2.1), we obtain a quasilinear system of five equations for determining the unknown functions σ, τ, Ψ, u, v, in the form

$$\begin{aligned} &\frac{\partial \sigma}{\partial r} + \cos 2\Psi \frac{\partial \tau}{\partial r} + \sin 2\Psi \frac{\partial \tau}{\partial z} - \\ &\quad 2\tau\left(\sin 2\Psi \frac{\partial \Psi}{\partial r} - \cos 2\Psi \frac{\partial \Psi}{\partial z}\right) = \frac{f_1}{r} \\ &\frac{\partial \sigma}{\partial z} + \sin 2\Psi \frac{\partial \tau}{\partial r} - \cos 2\Psi \frac{\partial \tau}{\partial z} + \\ &\quad 2\tau\left(\cos 2\Psi \frac{\partial \Psi}{\partial r} + \sin 2\Psi \frac{\partial \Psi}{\partial z}\right) = \frac{f_2}{r} \\ &f_1 = \frac{2k}{B} + \frac{2A+B}{B}\sigma - \left(\cos 2\Psi + \frac{2C}{B}\right)\tau, \quad f_2 = -\tau \sin 2\Psi \end{aligned} \tag{2.3}$$

$$\begin{aligned} &\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z} - \frac{2A}{B}\frac{u}{r} = 0, \quad \frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} - \frac{2C}{B}\frac{u}{r}\sin 2\Psi = 0 \\ &\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right)\cos 2\Psi - \left(\frac{\partial u}{\partial r} - \frac{\partial v}{\partial z}\right)\sin 2\Psi = 0 \end{aligned} \tag{2.4}$$

We should consider, together with relations (2.4), the conditions of associability of the principal velocities of deformation with the principal stresses, and this leads to the need for the following functional and differential inequalities to hold:

$$u < 0, \quad \left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right)^2 < 4\frac{\partial u}{\partial r}\frac{\partial v}{\partial z} \tag{2.5}$$

for the sides $Q_{1\varphi}{}^+Q_{12}{}^-$, $Q_{12}{}^-Q_{2\varphi}{}^+$ and of the inequality of oppostie sign for the sides $Q_{2\varphi}{}^-Q_{1\varphi}{}^+$, $Q_{2\varphi}{}^+Q_{1\varphi}{}^-$, $Q_{1\varphi}{}^-Q_{12}{}^+$ and $Q_{2\varphi}{}^+Q_{1\varphi}{}^-$.

The system (2.3), (2.4) which follows directly from the associated flow of law with condition (1.2), is not subject to classification /9/. The characteristic determinant of (2.3), (2.4) is identically equal to zero, since the matrices accompanying the derivatives in r and z of the functions required, are degenerate. Therefore, we can assume that separate differential relations exist, which follow from (2.2), for u, v and their derivatives and for the function Ψ and its derivatives. The relation between the velocities u, v is established by eliminating $\cos 2\Psi, \sin 2\Psi$ from (2.4), and this leads to the following non-linear hyperbolic system:

$$\left(\frac{\partial u}{\partial r} - \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right)^2 = \left(\frac{2C}{B}\frac{u}{r}\right)^2, \quad \frac{\partial u}{\partial r} + \frac{\partial v}{\partial z} = \frac{2A}{B}\frac{u}{r} \tag{2.6}$$

with characteristics

$$dz/dr = \operatorname{tg} \Psi, \quad dz/dr = -\operatorname{ctg} \Psi \tag{2.7}$$

Relations (2.7) determine, in the r, z plane, the principal directions of the stress tensor (they are coaxial with, and therefore also represent the principal trajectories of the deformation rate tensor). The corresponding corollaries of system (2.6) for the Coulomb-Mohr condition ($\alpha = \sin \varphi^\circ$, $\beta = 0$), and the Tresca-Saint Venant condition ($\alpha = \beta = 0$) were derived in /2, 6/. However, the formulation of the boundary value problems for system (2.6) presupposes kinematic definiteness, and this cannot be realized in the course of solving specific examples of axisymmetric problems in the rigid-plastic formulation. It is therefore important to establish the relationship for the function $\Psi (r, z)$ and its derivatives.

Let us turn our attention to relations (2.2), which yield

$$\frac{\partial u}{\partial r}=\frac{u}{r}\left(\frac{A}{B}+\frac{C}{B}\cos 2\Psi\right), \quad \frac{\partial v}{\partial z}=\frac{u}{r}\left(\frac{A}{B}-\frac{C}{B}\cos 2\Psi\right) \tag{2.8}$$

Therefore we can write

$$u=\Phi(r,z)\,r^{\delta}, \quad v=r^{\delta}\int\left(\frac{\delta\Phi}{r}-\frac{\partial\Phi}{\partial r}\right)dz+g(r), \quad \delta=\frac{A}{B} \tag{2.9}$$

$$\Phi(r,z)=f(z)\exp\left(\frac{C}{B}\int\frac{\cos 2\Psi}{r}\,dr\right) \tag{2.10}$$

where $f(z)$, $g(r)$ are arbitrary functions of their arguments from C^1.

If we now substitute the explicit expressions for the velocities of displacement (2.8) into the second equation of (2.1) and carry out a series of consecutive transformations, we obtain the following equation for the function (2.10):

$$\frac{\partial^2\Phi}{\partial z^2}+G\frac{\partial^2\Phi}{\partial r\,\partial z}-\frac{\partial^2\Phi}{\partial r^2}-E\frac{\partial\Phi}{\partial z}+\delta(\delta-1)\frac{\Phi}{r^2}=0 \tag{2.11}$$

$$G=\frac{4r}{F}\frac{\partial\Phi}{\partial r}, \quad E=\frac{4C}{B}\frac{\Phi}{rF}. \quad F=\left[\left(\frac{2C}{B}\Phi\right)^2-\left(2r\frac{\partial\Phi}{\partial r}\right)^2\right]^{1/2}$$

Note that (2.10) yields

$$\Psi(r,z)=\frac{1}{2}\arccos\left(\frac{B}{C}\frac{r}{\Phi}\frac{\partial\Phi}{\partial r}\right), \quad G=2\operatorname{ctg}2\Psi, \quad E=\frac{2}{r\sin 2\Psi} \tag{2.12}$$

Using the standard procedure for investigating second-order equations /10/, we find that the quasilinear Eqs. (2.11) is hyperbolic and its characteristics are the principal directions of the stress tensor in the r, z plane (2.7). We shall call (2.10), in accordance with (2.12), the function of characteristic directions.

Thus the degenerate system of Eqs. (2.3), (2.4) is transformed identically to: Eq. (2.11) for determining the function $\Phi(r,z)$ and hence $\Psi(r,z)$; system (2.3) linear with respect to the functions σ, τ, and system (2.5) non-linear with respect to the function u, v (the latter can be linearized trivially when the function Ψ is known, or when relations (2.9) are used to determine the displacement rates).

Let us further introduce, in the r, z plane, a curvilinear orthogonal system of coordinates coupléd to the principal directions of the stress tensor (2.9) $r=r(\alpha,\beta)$, $z=z(\alpha,\beta)$ (not to be confused with the constants α, β of the medium in Sect.1).

Also let the first equation of (2.7) define the α line ($\beta=\text{const}$) and the second equation the β line ($\alpha=\text{const}$). In this case the differentiation operators along the α-, β-lines will take the form

$$\frac{d}{dr_\alpha}=\frac{\partial}{\partial r}+\operatorname{tg}\Psi\frac{\partial}{\partial z}, \quad \frac{d}{dr_\beta}=\frac{\partial}{\partial r}-\operatorname{ctg}\Psi\frac{\partial}{\partial z} \tag{2.13}$$

Changing in (2.3), (2.4) to differentiation along the characteristics we obtain, in accordance with (2.13),

$$\frac{d(\sigma+\tau)}{dr_\alpha}=2\tau\operatorname{tg}\Psi\frac{d\Psi}{dr_\beta}+\frac{f_1+\operatorname{tg}\Psi f_2}{r} \tag{2.14}$$

$$\frac{d(\sigma-\tau)}{dr_\beta}=2\tau\operatorname{ctg}\Psi\frac{d\Psi}{dr_\alpha}+\frac{f_1-\operatorname{ctg}\Psi f_2}{r}$$

$$\frac{dU}{dr_\alpha}-V\frac{d\Psi}{dr_\alpha}=\frac{U\cos\Psi-V\sin\Psi}{r}\frac{A+C}{B\cos\Psi} \tag{2.15}$$

$$\frac{dV}{dr_\beta}+U\frac{d\Psi}{dr_\beta}=\frac{U\cos\Psi-V\sin\Psi}{r}\frac{A-C}{B\cos\Psi}$$

(U, V are the rates of displacement along the characteristics (2.7), respectively). We must supplement relations (2.15) along the trajectories of maximum elongation with the condition that the rate of shear is equal to zero

$$\sin\Psi\frac{dU}{dr_\beta}+\frac{V}{R_\beta}+\cos\Psi\frac{dV}{dr_\alpha}+\frac{U}{R_\alpha}=0 \tag{2.16}$$

where R_α, R_β are the radii of curvature of the α- and β-lines.

All this implies that the constructive algorithm for solving axisymmetric problems for a rigid-plastic body following the sides of the piecewise smooth condition (1.1) can be reduced, essentially, to solving the boundary value problems for Eq. (2.11), after which systems (2.14) and (2.15) can be regarded as relations along the characteristics of the stressed and deformed state, and determine the components of the stress and displacement rate tensors.

3. Formulating the boundary value problems for Eq.(2.11). Let us consider the possibility of formulating a boundary value problem for the equation in terms of the function Φ of characteristic stresses, starting by defining, on a certain curve in the r, z plane (Fig.2) the stresses, displacement rates, and their combinations. We shall assume that the curve ab is given in parametric form: $r = r(t),\ z = z(t);\ t \in [t_1, t_2]$, and $r(t), z(t) \in C^1_{[t_1, t_2]}$, and we shall denote the values of the functions on ab by the superscript $^\circ$. Let us consider, without loss of generality, the following three cases of formulating the problems which reduce to the Cauchy problem for (2.11).

1°. The components of the stress tensor are specified on the arc ab

$$\sigma_n|_{ab} = \sigma_n^\circ,\quad \sigma_s|_{ab} = \sigma_s^\circ,\quad \tau_{ns}|_{ab} = \tau_{ns}^\circ \tag{3.1}$$

Then (a prime denotes a derivative in t)

$$\Psi^\circ = \eta + {}^1/_2 \operatorname{arctg} 2\tau_{ns}^\circ/(\sigma_n^\circ - \sigma_s^\circ);\ \operatorname{tg}\eta = -r'(t)/z'(t) \tag{3.2}$$

We obtain $\Phi|_{ab}$ from (2.10) as

$$\Phi^\circ(t) = f^\circ(t)\exp\left(\frac{C}{B}\int\frac{\cos 2\Psi^\circ}{r(t)}r'(t)\,dt\right) \tag{3.3}$$

Then from (2.10) and the derivative with respect to t of (3.3) we obtain

$$\left.\frac{\partial\Phi}{\partial r}\right|_{ab} = \frac{C}{B}\frac{\Phi^\circ}{r}\cos 2\Psi^\circ,\quad \left.\frac{\partial\Phi}{\partial z}\right|_{ab} = \frac{df^\circ}{dt}\frac{\Phi^\circ}{f^\circ} \tag{3.4}$$

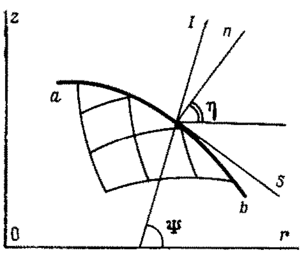

Fig.2

2°. The displacement rate components are specified on ab

$$u|_{ab} = u^\circ,\ v|_{ab} = v^\circ \tag{3.5}$$

Using Eqs.(2.2) we differentiate u°, v° along ab to obtain

$$\cos 2(\Psi^\circ - \eta) = \frac{A}{C} + \frac{Br}{Cu^\circ}\times(v^{\circ\prime}\cos\eta - u^{\circ\prime}\sin\eta) \tag{3.6}$$

Therefore the function Φ and its first derivatives on ab are given by the relations

$$\Phi|_{ab} = u^\circ r^{-\delta},\quad \left.\frac{\partial\Phi}{\partial r}\right|_{ab} = \frac{C}{B}u^\circ r^{-(\delta-1)}\cos 2\Psi^\circ$$
$$\left.\frac{\partial\Phi}{\partial z}\right|_{ab} = \frac{r^{-\delta}r'}{z'}\left[\frac{u^{\circ\prime}}{z'} - \frac{u^\circ}{Br}(A + C\cos\Psi^\circ)\right],\quad \delta = \frac{A}{B} \tag{3.7}$$

3°. The following mixed conditions are specified on ab:

$$\sigma_n|_{ab} = \sigma_n^\circ,\ \tau_{ns}|_{ab} = \tau_{ns}^\circ,\quad u|_{ab} = u^\circ \tag{3.8}$$

From (2.9) and differentiation of u° along ab we obtain

$$\cos 2\Psi^\circ = \frac{B}{C}\frac{r}{r'}\left(\frac{u^{\circ\prime}}{u^\circ} - \frac{r'}{Br} - \frac{f^{\circ\prime}}{f^\circ}\right) \tag{3.9}$$

and hence the values of the function Φ and its derivatives along the arc ab in the form

$$\Phi|_{ab} = u^\circ r^{-\delta},\quad \left.\frac{\partial\Phi}{\partial z}\right|_{ab} = \frac{u^\circ r^{-\delta}}{z'}\frac{f^{\circ\prime}}{f^\circ}$$
$$\left.\frac{\partial\Phi}{\partial r}\right|_{ab} = u^\circ r^{-\delta}\left(\frac{u^{\circ\prime}}{u^\circ} - \frac{\delta r'}{r} - \frac{f^{\circ\prime}}{f^\circ}\right) \tag{3.10}$$

The values of the functions f° and its derivative in expressions (3.4) and (3.10) are determined in terms of the stresses (3.1) and (3.8). Moreover, the right-hand sides of Eqs. (3.6) and (3.9) must not exceed unity in their absolute values, and this imposes restrictions, in the form of inequalities, on the velocities u°, v° and their first derivatives. In the case when the arc ab has characteristic directions $(\Psi^\circ = \eta)$, we obtain from (3.2), (3.6) and (3.9) relations for the initial conditions along the characteristics.

Thus, using relations (3.3), (3.4), (3.7) and (3.10), we arrive at the Cauchy problem for Eq.(2.11). The theorems of uniqueness and existence of solutions of this problem for a quasi-linear second-order equation were given in /10/.

4. Defining equations for an edge. Following Fig.1, we shall consider the edges $Q_{1\varphi}^+$, $Q_{1\varphi}^-$, $Q_{2\varphi}^+$ and $Q_{2\varphi}^-$, which have the corresponding relations (1.5) in the space of stresses $\sigma_r, \sigma_z, \sigma_\varphi, \tau_{rz}$. Using (1.5) as the plastic potential, we will have

$$\varepsilon_r = \xi + \zeta\cos 2\Psi,\ \varepsilon_z = \xi - \zeta\cos 2\Psi,\ \varepsilon_\varphi = \lambda B + \mu B_1 \tag{4.1}$$

$$\gamma_{rz} = 2\zeta \sin 2\Psi; \ \xi = \lambda A + \mu A_1, \ \zeta = \lambda C + \mu C_1, \ \lambda, \mu > 0$$

A, A_1, B, B_1, C, C_1 are found from (1.3) and (1.4) as the coefficients of the adjacent edges (Fig.1).

Changing in the equations of equilibrium and in (1.5) to the variables (2.1) and to the displacement rates in (4.1), we arrive at a quasilinear system of four equations for the functions σ, Ψ, u, v:

$$(1-d)\frac{\partial\sigma}{\partial r} - g\frac{\partial\sigma}{\partial z} - 2\tau\left(\sin 2\Psi \frac{\partial\Psi}{\partial r} - \cos 2\Psi \frac{\partial\Psi}{\partial z}\right) = \frac{\tau(\varkappa - \cos 2\Psi)}{r} \tag{4.2}$$

$$-g\frac{\partial\sigma}{\partial r} + (1+d)\frac{\partial\sigma}{\partial z} - 2\tau\left(\cos 2\Psi \frac{\partial\Psi}{\partial r} + \sin 2\Psi \frac{\partial\Psi}{\partial z}\right) = \frac{-\tau\sin 2\Psi}{r}$$

$$(\cos 2\Psi - \sin\chi)\frac{\partial u}{\partial r} + (\cos 2\Psi + \sin\chi)\frac{\partial v}{\partial z} = -\cos 2\Psi(1 - \varkappa\sin\chi)\frac{u}{r} \tag{4.3}$$

$$\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right)\cos 2\Psi - \left(\frac{\partial u}{\partial r} - \frac{\partial v}{\partial z}\right)\sin 2\Psi = 0$$

Here

$$d = \cos 2\Psi \sin\chi, \quad g = \sin 2\Psi \sin\chi, \quad \tau = K - \sigma\sin\chi$$
$$\sin\chi = (2\alpha + \beta)/(2 - \varkappa\beta), \quad K = 2k/(2 - \varkappa\beta)$$

Eqs.(4.3) must be supplemented by the inequality

$$\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right)^2 > 4\frac{\partial u}{\partial r}\frac{\partial v}{\partial z}, \quad Q_{1\varphi}^+, Q_{2\varphi}^+ \tag{4.4}$$

which is obtained by considering the signs of the principal deformation rates referred to the corresponding apices. In the case of the edges $Q_{1\varphi}^-, Q_{2\varphi}^-$ we must, in addition to (4.4), also have $u > 0$. The edges Q_{12}^+, Q_{12}^- (Fig.1) lead to the trivial state

$$\sigma_r = \sigma_z = \frac{2k_0}{\Delta+1} + \frac{C_0}{r^{(1+\Delta)}}, \quad \tau_{rz} = 0, \quad \sigma_\varphi = \frac{2k_0}{\Delta+1} + \frac{C_0\Delta}{r^{(1+\Delta)}} \tag{4.5}$$

$$C_0 = \text{const}, \quad \Delta = \frac{\alpha+\beta-\varkappa}{\alpha+\varkappa}, \quad k_0 = \frac{k}{\alpha+\varkappa}, \quad \varkappa = \begin{cases} -1, & Q_{12}^- \\ 1, & Q_{12}^+ \end{cases}$$

We have the following hyperbolic system for the velocities:

$$\frac{\partial u}{\partial r} + \frac{\partial v}{\partial z} - \frac{\Delta u}{r} = 0, \quad \frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} = 0 \tag{4.6}$$

with the characteristics $r = \text{const}$, $z = \text{const}$ and inequalities

$$u > 0, \quad \frac{\partial u}{\partial r} > 0, \quad \frac{\partial v}{\partial z} < 0, \quad (Q_{12}^+)$$
$$u < 0, \quad \frac{\partial u}{\partial r} > 0, \quad \frac{\partial v}{\partial z} > 0, \quad (Q_{12}^-)$$

Let us consider system (4.2) and (4.3). The system implies that we can consider, generally speaking, two equations of (4.2) independently of the last two equations of (4.3) (a statically determinate problem). However, in order to construct the matching stress and velocity fields while solving the specific problems, we must consider systems (4.2) and (4.3) simultaneously. Therefore henceforth we shall consider Eqs.(4.2) and (4.3) as a single quasilinear system in the functions σ, Ψ, u, v. The characteristic analysis shows that (4.2) and (4.3) is a hyperbolic system with double characteristics

$$dz/dr = \operatorname{tg}\gamma^{\alpha,\beta}, \quad \gamma^\alpha = \Psi - \chi/2 - \pi/4, \quad \gamma^\beta = \Psi + \chi/2 + \pi/4 \tag{4.7}$$

α corresponds to the α-line and β to the β-line.

Thus the characteristics of the stress field are identical to the characteristics of the velocity field.

Let us now define the differentiation along the characteristics

$$d^{\alpha,\beta}/dr = \partial/\partial r + \operatorname{tg}\gamma^{\alpha,\beta}\partial/\partial z$$

Then the relations along (4.7) can be written thus (the upper index refers to the α-line and the lower to the β-line):

$$d^{\alpha,\beta}\sigma \mp \frac{2\tau}{\cos\chi} d^{\alpha,\beta}\Psi = \frac{\tau}{r\cos\chi}[\varkappa\cos\chi\, d^{\alpha,\beta}r \mp (\varkappa\sin\chi - 1)\, d^{\alpha,\beta}z] \tag{4.8}$$

$$d^\alpha U - \left(U\operatorname{tg}\chi + \frac{V}{\cos\chi}\right)d^\alpha\Psi = \frac{\varkappa\sin\chi - 1}{2\sin\gamma^\alpha}\frac{u}{r}d^\alpha z \tag{4.9}$$

$$d^\beta V + \left(\frac{U}{\cos\chi} + V\operatorname{tg}\chi\right)d^\beta\Psi = \frac{\varkappa\sin\chi - 1}{2\sin\gamma^\beta}\frac{u}{r}d^\beta z$$

Here U, V are the rates of displacement along the α- and β-lines, respectively, (α-, β-characteristics are not orthogonal: $\gamma^\beta - \gamma^\alpha = \chi + \pi/2$).

5. Stress and displacement rate discontinuities. Let us investigate the possibility that a strong discontinuity in the stress and displacement rates exists in the neighbourhood of some curve Γ in the r, z plane. Let the stress state near Γ(Fig.3) correspond to a side of the condition (1.1) and let relations (1.2)-(1.4) hold. We shall introduce the local system of coordinates (n, s), associated with Γ as shown in Fig.3. Then the conditions of equilibrium of the element of the medium in the immediate neighbourhood of the proposed stress discontinuity curve Γ will require that the stress jumps $[\sigma_n] = [\tau_{ns}] = 0$. Here σ_s and σ_φ may be themselves discontinuous.

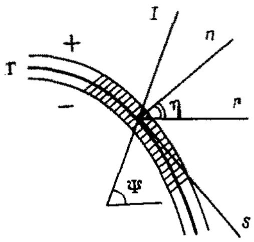

Fig.3

Let us write the components of the stress tensor in the n, s axes

$$\sigma_n = \sigma + \tau \cos 2 (\Psi - \eta), \qquad (5.1)$$
$$\sigma_s = \sigma - \tau \cos 2 (\Psi - \eta)$$
$$\tau_{ns} = \tau \sin 2 (\Psi - \eta)$$

The condition of plasticity (1.1) can be written in accordance with (2.1), in the form

$$\sigma_\varphi = 2B^{-1} (k - A\sigma - C\tau) \qquad (5.2)$$

Using the relations $[\sigma_n] = [\tau_{ns}] = 0$ we obtain an equation connecting the jumps Ψ, σ_φ and τ across Γ

$$C \cos (\Psi^+ - \Psi^- - 2\eta) - \frac{B}{4} \frac{[\sigma_\varphi]}{\tau^-} \frac{\sin 2 (\Psi^+ - \eta)}{\sin (\Psi^+ - \Psi^-)} = A \cos (\Psi^+ - \Psi^-) \qquad (5.3)$$

Thus, knowing the values of Ψ, τ on one side of Γ and being given the jump in the value of the peripheral stress $[\sigma_\varphi]$, we can find Ψ^+ and hence τ^+, σ^+ on the other side of Γ.

If we assume that σ_φ remains continuous during the passage across Γ ($[\sigma_\varphi] = 0$), then (5.3) simplifies and the jump $[\Psi]$ will be independent of τ and hence also of σ. The same result is obtained when (5.2) is studied on both sides of Γ.

Now let Γ be a part of the characteristic, i.e. $\eta = \Psi$ on a segment of finite length. Now (5.3) loses its meaning. Let us turn our attention to the relations on the characteristics (2.14)-(2.16). This shows at once that $[\Psi] = 0$, σ_1 ($[\sigma_2] = 0$) can be discontinuous along the first principal direction (α–line) and σ_2 ($[\sigma_1] = 0$) along the second direction (β–line). We have the following relations for the jumps:

$$[\sigma_1] = -B (A + C)^{-1} [\sigma_\varphi] \ \text{(along the } \alpha\text{-line)} \qquad (5.4)$$
$$[\sigma_2] = B (C - A)^{-1} [\sigma_\varphi] \ \text{(along the } \beta\text{-line)}$$

Next we consider the possibility of the jumps in the displacement rates near the curve Γ in the r, z plane, using relations (2.2) in which r and z are replaced by n and s. This is based on the fact that (2.2) is invariant under the choice of the rectangular system of coordinates in the r, z plane, from which we have

$$\frac{\varepsilon_s - \varepsilon_n}{\sigma_s - \sigma_n} = \frac{C^2 (\varepsilon_s + \varepsilon_n)}{A [2k - A (\sigma_s + \sigma_n) - B\sigma_\varphi]} = \frac{\varepsilon_{sn}}{\tau_{sn}} = \frac{2\lambda C}{\sqrt{\Sigma}}; \quad \lambda \geqslant 0 \qquad (5.5)$$

Imposing on (5.5) the condition that the tangential velocity component is discontinuous along Γ ($\varepsilon_{ns}/\varepsilon_n \sim \varepsilon_{ns}/\varepsilon_s \to \infty$), we arrive at the equations

$$\sigma_s - \sigma_n = 0, \ A (\sigma_s + \sigma_n) + B\sigma_\varphi = 2k \qquad (5.6)$$

Therefore will find from the condition of plasticity (5.2) that $\sigma_1 = \sigma_2$, and we arrive either at the edges Q_{12}^+ or Q_{12}^- of the yield pyramid (4.5), (4.6).

It remains to consider the case when Γ contains the characteristic directions. From the relations along the characteristics (2.15) it follows that the tangential component of the displacement rate U may have a discontinuity across the α–line. The component V is discontinuous along the β–line. But, since the characteristics (2.7) are the trajectories of the principal directions of the deformation rates, the condition that the principal shear (2.16) be zero must hold along them. This imposes a constraint on the jumps in the velocity along the α- and β–lines.

Indeed, let there be a jump in the value of the tangential component $[U] \neq 0$ along the α–line. Then from (2.15) it follows necessarily that $[V] \neq 0$ and the radii of curvature R_α and R_β must be equal to each other ($R_\alpha = R_\beta$), and the jumps in the displacement rate components must satisfy the relations

$$[U] = -[V], \quad \text{or} \quad U^+ + V^+ = U^- + V^- \qquad (5.7)$$

Thus in the case of axial symmetry, when the deformation rates (2.2) correspond to the sides, the displacement rate may become discontinuous only along the characteristics (5.7) and both the normal and tangenti component of (5.7) will be discontinuous. Therefore the region of the medium with condition of plasticity (1.2) adjacent to the curve of discontinuity of the displacement rate component Γ should be regarded as a mathematical idealization of a transition layer of finite thickness. Thus, if Γ is a segment of a rigid-plastic boundary, then the introduction of such a layer can be justified by the dilatation effect (the change in volume related to the shear deformation intensity).

When $\alpha, \beta \neq 0$ simultaneously, we obtain the following linear relation from (2.2):

$$\varepsilon = \Lambda\gamma, \quad \Lambda = \frac{3}{2}(2\alpha+\beta)[(\alpha-\beta)^2+3]^{-1/2} \tag{5.8}$$
$$\varepsilon = \varepsilon_r + \varepsilon_z + \varepsilon_\varphi,$$
$$\gamma = \frac{1}{3}\left\{2\left[(\varepsilon_r-\varepsilon_z)^2+(\varepsilon_z-\varepsilon_\varphi)^2+(\varepsilon_\varphi-\varepsilon_r)^2+\frac{3}{2}\gamma_{rz}^2\right]\right\}^{1/2}$$

Here the passage from the non-deformed part of the medium to the medium with (5.8), is made across a layer of finite thickness with the discontinuity properties described above.

Let us now investigate the stressed and deformed state corresponding to the edges of the yield condition (1.5). Let a strong discontinuity exist in the tensor components near the curve Γ (Fig.3) and $[\sigma_n] = [\tau_{ns}] = 0$, $[\sigma_s] \neq 0$. We should supplement relations (5.1) with the equations

$$A\sigma + C\tau = 2k, \quad \sigma_\varphi = \sigma + \varkappa\tau \tag{5.9}$$

and this yields, after transforming (5.3),

$$\cos(\Psi^+ - \Psi^-) = A^{-1}C\cos(\Psi^+ - \Psi^- - 2\eta) \tag{5.10}$$

Specifying on Γ the jump $[\Psi]$ and the values of σ and τ on one side of the discontinuity curve, we can determine, using (5.10), $[\tau]$ and $[\sigma]$. From the relations for the stresses (4.8) along the characteristics it follows that $[\sigma] = [\Psi] = 0$ during passage through the characteristic. Then by virtue of (5.9) we also have $[\tau] = [\sigma_\varphi] = 0$. Therefore, when the state of stress corresponds to the edges (1.5), the stress tensor components are continuous along the characteristics (4.7).

Let us now pause and consier the discontinuities in the displacement rate components along the characteristics (4.7). We find from (4.9) that $[U] \neq 0$ across the α-line, and for the β-line we have $[V] \neq 0$.

To be specific, we shall consider the α-line and assume that near this line u is directed along the tangent, i.e. it coincides with U and $V = 0$ ($\Psi - \pi/4 - \chi/2 = 0 \Rightarrow \Psi = \pi/4 + \chi/2$). Then $u = U$, $v = u \operatorname{tg} \chi$. This implies that in the case of a medium with friction corresponding to the edge of the generalized condition (1.1), the jump in the tangential component of the velocity leads also to a jump in the value of the normal component, and $[v] = [u] \operatorname{tg} \chi$. Thus, in this case, when the velocities are discontinuous along the characteristics, we ought to assume that a transition layer of finite thickness exists in the neighbourhood of such a characteristic. If $\chi = 0$ ($\alpha = \beta = 0$), we have an edge of the Tresca plasticity prism, the normal component of the displacement rate has no discontinuity and the necessity no longer applies in the transition layer (it becomes a line).

We note that the special feature of occurrence of a simultaneous discontinuity in the normal and tangential component of the displacement rate for the media with yield condition depending on the mean pressure, was pointed out in /11/. Finally, carrying out the general argument as in /12/, we conclude that the line separating the rigid region from the adjacent plasticity deformable region represents a characteristic for both the side as wll as the edge of condition (1.1).

6. Dissipation function. Variational approach. Let us determine the function of mechanical energy dissipation per unit volume of the medium deformed according to the law (2.2). We shall use the repesentation of the stress and deformation rate tensors for the axisymmetric case in the form of five-dimensional vectors $\boldsymbol{\varepsilon}, \boldsymbol{\sigma}$. Then in accordance with the definition of the dissipation function D, the scalar product $\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}$, i.e.

$$D = \sigma_{ij}\varepsilon_{ij} = |\sigma|\cdot|\varepsilon|\cos(\sigma,\varepsilon) = 2\lambda k \quad (i, j = r, z, \varphi) \tag{6.1}$$
$$\varepsilon_{ij}\varepsilon_{ij} = |\varepsilon|^2 = \lambda^2(2\alpha^2+\beta^2+2); \quad 2\varepsilon_{rz} = \gamma_{rz}$$

This yields

$$D = k_0\left(\varepsilon_r^2+\varepsilon_z^2+\varepsilon_\varphi^2+\frac{1}{2}\gamma_{rz}^2\right)^{1/2} = k_0|\varepsilon| \tag{6.2}$$
$$|\sigma|\cos(\sigma,\varepsilon) = k_0; \quad k_0 = 2k(2\alpha^2+\beta^2+2)^{-1/2}$$

The equation of the rate of virtual work when the rate change is vanishingly small can be written in the form

$$\int\limits_{\Omega} D\,d\omega = \int\limits_{\partial\Omega}(P_r u + P_z v)\,ds \tag{6.3}$$

where Ω is the region of plastic deformation in the r, z plane with the boundary $\partial\Omega$, P_r, P_z are the stress distribution densities on $\partial\Omega$.

We assume in (6.3), for simplicity, that there are no volume forces in Ω and, that the u, v velocity field is smooth.

Below we shall introduce the statistically possible stresses σ_{ij}^* and kinematically admissible deformation rates ε_{ij}° /5-7, 12/, and denote by σ_{ij}, ε_{ij} the real values of the stresses and deformation rates corresponding to relations (2.2) and satisfying the boundary conditions on $\partial\Omega = \partial\Omega_p \cup \partial\Omega_V$ (the stresses are specified on $\partial\Omega_p$ and deformation rates on $\partial\Omega_V$). From the maximum principle for the dissipation function we have

$$(\sigma_{ij}^* - \sigma_{ij})\,\varepsilon_{ij} \leqslant 0 \tag{6.4}$$

Then from (6.3) and (6.4) we obtain

$$\int_{\partial\Omega_V} (P_r^* u + P_z^* v)\, ds \leqslant \int_{\partial\Omega_V} (P_r u + P_z v)\, ds \tag{6.5}$$

which corresponds to the maximum power developed by the real surface forces as compared with any other statistically possible system P_r^*, P_z^*.

We will now derive the analogue of the minimum of the real velocity field for a rigid-plastic medium (2.2), assuming that the real distribution of the stresses σ_{ij} satisfies, at every point of the deformable body, the inequality (see the second equation of (6.2))

$$|\,\sigma_{ij}\,| \leqslant K_0 < \infty \tag{6.6}$$

where $K_0 = \text{const}$ is found in terms of the physical strength parameters k, α, β.

Let us now turn to (6.2) and consider the scalar product

$$\sigma_{ij}(\varepsilon_{ij}^\circ - \varepsilon_{ij}) \leqslant |\sigma|\,|\varepsilon^\circ| - k_0|\varepsilon| < k_0(T_0|\varepsilon^\circ| - |\varepsilon|), \quad T_0 = K_0/k_0 \tag{6.7}$$

Further, applying (6.3) to $\sigma_{ij}(\varepsilon_{ij}^\circ - \varepsilon_{ij})$ and taking into account (6.7), we obtain

$$k_0 T_0 \int_\Omega |\varepsilon_{ij}^\circ|\, d\omega - \int_{\partial\Omega_p} (P_r u^\circ + P_z v^\circ)\, ds > \tag{6.8}$$

$$k_0 \int_\Omega |\varepsilon_{ij}|\, d\omega - \int_{\partial\Omega_p} (P_r u + P_z v)\, ds = \int_{\partial\Omega_V} (P_r u + P_z v)\, ds$$

The inequalities (6.5) and (6.8) yield a two-sided estimate for the power of the real surface forces for the given velocities. Moreover, using (6.8) we can place Eq. (6.3) in 1:1 correspondence with the problem of determining the minimum of the functional

$$J(u, v) = M(u, v) + L(u, v) \tag{6.9}$$

in the linear space

$$H(\Omega) = \{u_0, v_0 \mid u_0 = u_1^\circ - u_2^\circ,\ v_0 = v_1^\circ - v_2^\circ\}$$

where (u_1°, v_1°), (u_2°, v_2°) are arbitrary, kinematically admissible pairs of displacement rate vectors of the points belonging to the volume Ω with the boundary $\partial\Omega_V \cup \partial\Omega_p$. We obtain $M(u, v)$, $L(u, v)$ in (6.9) thus:

$$M(u, v) = k_0 T_0 \int_\Omega \left[\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial v}{\partial r}\right)^2 + \left(\frac{u}{r}\right)^2\right]^{1/2} d\omega \tag{6.10}$$

$$L(u, v) = \int_{\partial\Omega} (P_r u + P_z v)\, ds$$

Following /13/, we establish that (6.10) is a convex non-differentiable functional, and some of the results of /13/ referring to the rigid-plastic body can be reformulated for (6.9). We note that when $\alpha \neq 0$, $\beta \neq 0$, the u, v velocity field is not solenoidal with respect to (5.8), and it therefore makes not sense to change to the relation connecting the stress and deformation rate deviators.

The hyperbolic second-order Eqs. (2.13) for the function $\Phi(r, z)$ of characteristic directions is reduced to a quasilinear first-order ststem /10/. In this case the boundary conditions (3.3)-(3.4), (3.7), (3.10) can also be reformulated in terms of the functions of such a system. A solution of the boundary-value problems for quasilinear hyperbolic systems, taking into account the possible discontinuities, can be constructed using the difference schemes of the type given in /14/ and similar ones.

It should also be noted that the mechanical energy dissipation function per unit volume of D (6.2) can be expressed linearly in terms of the scalar function of characteristic directions $\Phi(r, z)$ (2.10) thus:

$$D = 2kr^{\delta-1}\,|\,\Phi(r, z)\,/\,B\,|$$

Then the problem of determining the minimum of the functional (6.9) can be formulated for Φ. We must introduce here in a proper manner the linear space $\mathbf{H}_\Phi(\Omega)$, and this may prove a decisive factor in proving the existence of a minimizing element in the axisymmetric problem (6.9).

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A GENERALIZATION OF THE CANONICAL FORM OF POINCARÉ'S EQUATIONS*

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A class of non-linear reversible replacements of canonical momenta is described, which reduces the Hamiltonian system to a form which differs only slightly from Poincaré's equations /1/ in canonical form, obtained by Chetayev /2/. The difference is solely the fact that the components of the operators which form the right-hand side of the equations of motion may depend on new variables (the Chetayev variables). The usual canonical form of the equations is obtained if the resplacements of the momenta are linear and uniform. Among the important consequences of the equations are Liouville's theorem (on complete integrability), the Kozlov-Kolesnikov theorem (on integrability in integral manifolds) /3/, and the theorem on classes of equivalence of Hamiltonian systems.

1. Initial data and relations.

Consider s continuously differentiable functions of the coordinates and canonical momenta

$$y_i = \psi_i(x, p),\ i = 1, \ldots, s \tag{1.1}$$

which are functionally independent and uniquely solvable (in a certain region) in terms of the variables p, i.e., $\det(\partial\psi_i/\partial p_j) \neq 0,\ p_j = \varphi_j(x, y)$ (the functions φ_j, naturally, are not defined everywhere), and generate an s-dimensional Lie algebra ($(\cdot,\cdot)$ are Poisson brackets)

$$(\psi_i, \psi_j) = c_{ij}{}^k\psi_k\quad i, j, k = 1, \ldots, s \tag{1.2}$$

Using the operators

$$X_k = \zeta_{x_i}^k \frac{\partial}{\partial x_i} + \zeta_{p_i}^k \frac{\partial}{\partial p_i};\quad \zeta_{x_i}^k = \frac{\partial\psi_k}{\partial p_i},\quad \zeta_{p_i}^k = -\frac{\partial\psi_k}{\partial x_i} \tag{1.3}$$

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